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Guidelines of the Answer

Subject : Linear Algebra

Semester : II

Department : Mathematics Education

1. $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda(3-\lambda)(3-\lambda) \rightarrow \lambda_1 = 0$ and $\lambda_{2,3} = 3$

for $\lambda = 0 \rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\lambda = 3 \rightarrow X_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Apply gram-schmidt process to X_2 and X_3 :

$W_2 = X_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $W_3 = X_3 - \frac{X_3.W_2}{W_2.W} W_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

The orthogonal basis are :

$W_1 = X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $W_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $W_3 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$, then normalized:

$W_1 \rightarrow Y_1 = \frac{W_1}{\| W_1 \|} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$, $W_2 \rightarrow Y_2 = \frac{W_2}{\| W_2 \|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, and $W_3 \rightarrow Y_3 = \frac{W_3}{\| W_3 \|} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$

So, the orthogonal matrix **P** is:

$\mathbf{P} = (W_1\ W_2\ W3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \rightarrow \mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{T}\mathbf{AP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

2. Transformation $T : R^5 \rightarrow R^4$, where : $T\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = \begin{pmatrix} x-2y-z+3s-2t \\ 3x-6y-2z+7s-t \\ -2x+4y+z-4s-t \\ 5x-10y-3z+11s \end{pmatrix}$.

(a) basis of image space Im(T) = { $\begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$ }, and dim Im(T) = 2.

(b) basis of null space Ker(T) = { $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}$ |, and dim Ker(T) = 3.

3. (a) Given : **A** similar **B**

$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

$\mathbf{B}^{-1} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{-1} = \mathbf{P}^{-1}\mathbf{A}^{-1}\mathbf{P} \rightarrow$ so, $\mathbf{A}^{-1}$ similar $\mathbf{B}^{-1}$.

(b) Given : ***A*** and ***B*** are similar,

$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

$|\boldsymbol{B} - \lambda\mathbf{I}| = |\mathbf{P}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}| = |\mathbf{P}^{-1}|\ |(\mathbf{A} - \lambda\mathbf{I})|\ |\mathbf{P}| = |\mathbf{A} - \lambda\mathbf{I}|$

So, ***A*** and ***B*** have same characteristics root (eigen value)!

4. $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -2-\lambda & 2 & 3 \\ -2 & 3-\lambda & 2 \\ -4 & 2 & 5-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda) \rightarrow \lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$

for $\lambda = 1 \rightarrow X_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, for $\lambda = 2 \rightarrow X_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}$, and for $\lambda = 3 \rightarrow X_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$. Because matrix ***A*** have three invariant vectors which are independent, so the matrix ***A*** is diagonalizable. Take matrix **P** = $(X_1\ X_2\ X_3) = \begin{pmatrix}1&1&1\\0&2&1\\1&0&1\end{pmatrix}$, then $\mathbf{P}^{-1} = \begin{pmatrix}2&-1&1\\1&0&-1\\-2&1&2\end{pmatrix} \rightarrow \mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$.

5. Basis orthogonal : $Y_1 = \begin{pmatrix}-1\\0\\0\\-1\end{pmatrix}$, $\mathbf{Y_2} = \begin{pmatrix}-\frac{1}{2}\\1\\0\\\frac{1}{2}\end{pmatrix}$, $\mathbf{Y_3} = \begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\1\\-\frac{2}{3}\end{pmatrix}$, and $\mathbf{Y_4} = \begin{pmatrix}\frac{4}{7}\\\frac{4}{7}\\-\frac{8}{7}\\-\frac{4}{7}\end{pmatrix}$.

   Basis orthonormal: $\mathbf{g_1} = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\0\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$, $\mathbf{g_2} = \begin{pmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\0\\\frac{1}{\sqrt{6}}\end{pmatrix}$, $\mathbf{g_3} = \begin{pmatrix}\frac{2}{\sqrt{21}}\\\frac{2}{\sqrt{21}}\\\frac{3}{\sqrt{21}}\\-\frac{2}{\sqrt{21}}\end{pmatrix}$, and $\mathbf{g_4} = \begin{pmatrix}\frac{1}{\sqrt{7}}\\\frac{1}{\sqrt{7}}\\-\frac{2}{\sqrt{7}}\\-\frac{1}{\sqrt{7}}\end{pmatrix}$.

6. Apply Gram-Schmidt process to X_1 and X_2 :

   $Y_1 = X_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$, and $Y_2 = X_2 - \frac{X_2.Y_1}{Y_1.Y_1} Y_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$. Normalization to vector Y_1 and Y_2, to be $g_1 = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{pmatrix}$ and $g_2 = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$. Finding orthonormal basis $\{g_1, g_2, g_3\}$, take the third vector $g_3 = \begin{pmatrix}a\\b\\c\end{pmatrix}$ is orthogonal to g_1 and g_2, and g_3 have norm equal 1, thus :

   $g_1 . g_3 = 0 \rightarrow \frac{a}{\sqrt{3}} + \frac{b}{\sqrt{3}} - \frac{c}{\sqrt{3}} = 0$

   $g_2 . g_3 = 0 \rightarrow \frac{a}{\sqrt{2}} + 0 + \frac{c}{\sqrt{2}} = 0$

   $||g_3|| = 1 \rightarrow a^2 + b^2 + c^2 = 1$

   From the system of equation above, $a = \frac{1}{6}\sqrt{6}$, $b = \frac{-2}{6}\sqrt{6}$, and $c = \frac{-1}{6}\sqrt{6}$; thus $g_3 = \begin{pmatrix}\frac{\sqrt{6}}{6}\\\frac{-2\sqrt{6}}{6}\\\frac{-\sqrt{6}}{6}\end{pmatrix}$.

7. If relative coordinate **q** to the basis **P** is $\mathbf{q}_P = (x, y, z)$, then :
   $\mathbf{q} = xp_1 + yp_2 + zp_3$
   $-1 - 3t + 3t^2 = x(1 + 2t + t^2) + y(3 + 8t - 2t^2) + z(2 + 5t)$
   It can get system of equation:
   $x + 3y + 2z = -1$
   $2x + 8y + 5z = -3$
   $x - 2y = 3$
   from the system above, $x = -1$, $y = -2$, and $z = 3$, thus the relative coordinate of **q** to the basis **P** is (-1, -2, 3).

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