Mata Kuliah : Linear Algebra

Rambu-rambu jawaban Mid Test, April 2011

1. (i) basis and dimension of ***U***,

a + c – d = 0, in matrix :

(1 0 1 -1 0) , because n = 5 and rank = 1, there are 4 free variables, that are : b, c, d, and e.

If b = 1, c = 0, d = 0, e = 0, then a = 0 → $(0, 1, 0, 0, 0)^T$

If b = 0, c = 1, d = 0, e = 0, then a = - 1 → $(-1, 0, 1, 0, 0)^T$

If b = 0, c = 0, d = 1, e = 0, then a = 1 → $(1, 0, 0, 1, 0)^T$

If b = 0, c = 0, d = 0, e = 1, then a = 0 → $(0, 0, 0, 0, 1)^T$

Basis of ***U*** = $\{ \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \}$ and dim ***U*** = 4.

(ii) basis and dimension of ***W***,

b + 3c – e = d

b = c – 2d; or

b + 3c – d – e = 0

b – c + 2d = 0; in matrix :

$\begin{pmatrix}0 & 1 & 3 & -1 & -1\\0 & 1 & -1 & 2 & 0\end{pmatrix}$ or in echelon form become $\begin{pmatrix}0 & 1 & 3 & -1 & -1\\0 & 0 & -4 & 3 & 1\end{pmatrix}$.

New equations :

b + 3c – d – e = 0

– 4c + 3d + e = 0; because n = 5 and rank 2, there are 3 free variables, that are : a, d, and e.

If a = 1, d = 0, e = 0, then c = 0, b = 0 → $(1, 0, 0, 0, 0)^T$

If a = 0, d = 1, e = 0, then c = 3/4, b = -5/4 → $(0, -5/4, 3/4, 1, 0)^T$

If a = 0, d = 0, e = 1, then c = 1/4, b = 1/4 → $(0, 1/4, 1/4, 0, 1)^T$

Basis of ***W*** = $\{ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-5/4\\3/4\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1/4\\1/4\\0\\1\end{pmatrix} \}$ or $\{ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-5\\3\\4\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\4\end{pmatrix} \}$ and dim ***W*** = 3.

(iii) basis and dimension of ***U*** ∩ ***W***,

a + c – d = 0

b + 3c – d – e = 0

b – c + 2d = 0; in matrix :

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & 1 & -1 & 2 & 0 \end{pmatrix} \text{ or in echelon form } \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & -4 & 3 & 1 \end{pmatrix}.$$

New equations :

a + c – d = 0

b + 3c – d – e = 0

-4c + 3d + e = 0; because n = 5 and rank = 3, there are 2 free variables, that are d and e.

If d = 1, e = 0, then c = ¾, b = -5/4, a = ¼ → $(1/4, -5/4, 3/4, 1, 0)^T$

If d = 0, e = 1, then c = ¼ , b = ¼ , a = - ¼ → $(-1/4, 1/4, 1/4, 0, 1)^T$

$$\text{Basis of } \boldsymbol{U} \cap \boldsymbol{W} = \{ \begin{pmatrix} 1/4 \\ -5/4 \\ 3/4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/4 \\ 1/4 \\ 1/4 \\ 0 \\ 1 \end{pmatrix} \} \text{ or } \{ \begin{pmatrix} 1 \\ -5 \\ 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 4 \end{pmatrix} \} \text{ and } \dim \boldsymbol{U} \cap \boldsymbol{W} = 2.$$

(iv) basis dimension of $\boldsymbol{U} + \boldsymbol{W}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -5 & 3 & 4 & 0 \\ 0 & 1 & 1 & 0 & 4 \end{pmatrix} \text{ or in echelon form } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Basis of } \boldsymbol{U} + \boldsymbol{W} = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \} \text{ and } \dim \boldsymbol{U} + \boldsymbol{W} = 5.$$

2. $x\boldsymbol{A} + y\boldsymbol{B} + z\boldsymbol{C} + t\boldsymbol{D} = \boldsymbol{0}$, diperoleh system persamaan linear homogen :

   6x + y – z – t = 0

   2x + y + z – t = 0

   -2x – y – z + t = 0

   -3x – 2y – 2z + t = 0

   Dari system tersebut diperoleh penyelesaian, misalnya t = 1, z = 2, y = -3, x = 1, yang berarti $\boldsymbol{A} - 3\boldsymbol{B} + 2\boldsymbol{C} + \boldsymbol{D} = \boldsymbol{0}$, yang berarti **A**, **B**, **C**, dan **D** adalah dependent (bergantung linear).

3. $\boldsymbol{p}_1 = x\boldsymbol{p}_2 + y\boldsymbol{p}_3 + z\boldsymbol{p}_4$, diperoleh system persamaan linear nonhomogen:
$3x - y + 2z = 1$
$8x - 3y + 5z = 2$
$-2x + 3y \quad = 1$
Dari system tersebut diperoleh penyelesaian : z = 3, y = -1, dan x = -2. Jadi kombinasi linearnya adalah $\boldsymbol{p}_1 = -2\boldsymbol{p}_2 - \boldsymbol{p}_3 + 3\boldsymbol{p}_4$

4. Bukti :

(i) ambil $\boldsymbol{O} \in \boldsymbol{U}$ dan $\boldsymbol{O} \in \boldsymbol{W}$, maka

$\boldsymbol{O} + \boldsymbol{O} = \boldsymbol{O}$, sehingga tampak bahwa $\boldsymbol{O} \in \boldsymbol{U} + \boldsymbol{W}$

Jadi $\boldsymbol{U} + \boldsymbol{W}$ tidak kosong

(ii) ambil $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{U}$ dan $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{W}$, berarti untuk sembarang skalar x dan y

$x\boldsymbol{u}_1, y\boldsymbol{u}_2 \in \boldsymbol{U}$ dan $x\boldsymbol{w}_1, y\boldsymbol{w}_2 \in \boldsymbol{W}$

$x(\boldsymbol{u}_1 + \boldsymbol{w}_1) + y(\boldsymbol{u}_2 + \boldsymbol{w}_2) = (x\boldsymbol{u}_1 + y\boldsymbol{u}_2) + (x\boldsymbol{w}_1 + y\boldsymbol{w}_2) \in \boldsymbol{U} + \boldsymbol{W}$

Jadi tertutup terhadap penjumlahan dan perkalian vektor.
Jadi terbukti bahwa $\boldsymbol{U} + \boldsymbol{W}$ adalah subspace dari ***V.***

5. Bukti :

Karena **u**, **v**, dan **w** bebas linear (*linearly independence*), maka hubungan

$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ **hanya berlaku** untuk a = b = c = 0. Sementara itu :

$x\mathbf{p} + y\mathbf{q} + z\mathbf{r} = \mathbf{0}$

$x(\mathbf{u} - \mathbf{v}) + y(\mathbf{u} + \mathbf{v}) + z(\mathbf{u} - 2\mathbf{v} + \mathbf{w}) = \mathbf{0}$

$(x + y + z)\mathbf{u} + (-x + y - 2z)\mathbf{v} + z\mathbf{w} = \mathbf{0}$

Karena **u**, **v**, dan **w** bebas linear (*linearly independence*), maka haruslah:

$x + y + z = 0$

$-x + y - 2z = 0$

$z = 0$

dari system persamaan linear homogen tersebut **hanya diperoleh** penyelesaian z = 0, y = 0, dan x = 0 atau :

$0\mathbf{p} + 0\mathbf{q} + 0\mathbf{r} = \mathbf{0}$, Jadi vektor-vektor **p**, **q**, dan **r** adalah *linearly independence*.

6. (a). $\boldsymbol{W} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x + y - 3z = 0 ;\ x, y, z \in \mathrm{R} \right\}$.

(i) $\boldsymbol{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \boldsymbol{W}$, sebab 2.0 + 0 – 3.0 = 0, jadi $\boldsymbol{W}$ tidak kosong.

(ii) ambil $\boldsymbol{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in \boldsymbol{W} \rightarrow 2x_1 + y_1 - 3z_1 = 0$

ambil $\boldsymbol{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \boldsymbol{W} \rightarrow 2x_2 + y_2 - 3z_2 = 0$

untuk sembarang scalar m dan n :

$$m\boldsymbol{u} + n\boldsymbol{v} = m\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + n\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} mx_1 \\ my_1 \\ mz_1 \end{pmatrix} + \begin{pmatrix} nx_2 \\ ny_2 \\ nz_2 \end{pmatrix} = \begin{pmatrix} mx_1 + nx_2 \\ my_1 + ny_2 \\ mz_1 + nz_2 \end{pmatrix}$$

$$2(mx_1 + nx_2) + (my_1 + ny_2) - 3(mz_1 + nz_2) =$$

$$= m(2x_1 + y_1 - 3z_1) + n(2x_2 + y_2 - 3z_2)$$

$$= m.0 + n.0 = 0$$

Jadi $m\boldsymbol{u} + n\boldsymbol{v} \in \boldsymbol{W}$ sebab $2(mx_1 + nx_2) + (my_1 + ny_2) - 3(mz_1 + nz_2) = 0$; yang berarti $\boldsymbol{W}$ tertutup terhadap penjumlahan dan perkalian.

Jadi $\boldsymbol{W}$ subspace dari $\boldsymbol{V}$.

(b). $\boldsymbol{W} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 3xyz = 0;\ x, y, z \in \mathrm{R} \}$.

(i) $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \boldsymbol{W}$, sebab 3.0.0.0 = 0, jadi $\boldsymbol{W}$ tidak kosong.

(ii) ambil $\boldsymbol{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in \boldsymbol{W} \rightarrow 3x_1y_1z_1 = 0$

ambil $\boldsymbol{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \boldsymbol{W} \rightarrow 3x_2y_2z_2 = 0$

untuk sembarang scalar m dan n :

$$m\boldsymbol{u} + n\boldsymbol{v} = m\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + n\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} mx_1 \\ my_1 \\ mz_1 \end{pmatrix} + \begin{pmatrix} nx_2 \\ ny_2 \\ nz_2 \end{pmatrix} = \begin{pmatrix} mx_1 + nx_2 \\ my_1 + ny_2 \\ mz_1 + nz_2 \end{pmatrix}$$

$3(mx_1 + nx_2)(my_1 + ny_2)(mz_1 + nz_2) = 3(m^3 x_1y_1z_1 + m^2n\, x_1y_1z_2 + m^2n\, x_1y_2z_1$
$+ mn^2\, x_1y_2z_2 + m^2n\, x_2y_1z_1 + mn^2\, x_2y_1z_2$
$+ mn^2\, x_2y_2z_1 + n^3\, x_2y_2z_2)$
$= 3(0 + m^2n\, x_1y_1z_2 + m^2n\, x_1y_2z_1$
$+ mn^2\, x_1y_2z_2 + m^2n\, x_2y_1z_1 + mn^2\, x_2y_1z_2$
$+ mn^2\, x_2y_2z_1 + 0) \neq 0$

Berarti m***u*** + n***v*** ≠ 0, tidak tertutup pada penjumlahan dan perkalian.
Jadi ***W*** bukan subspace dari ***V***.

7. Suppose ***V*** = $\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathrm{R}\}$. Addition and scalar multiplication operations on ***V*** are defined as follow :

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix} \text{ and } k\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ kc_1 \end{pmatrix}$$, for any scalar *k*.

Berdasarkan definisi tersebut, ***V*** bukan ruang vektor, sebab misalnya **tidak berlaku sifat distributive**.

oooo000oooSEMOGA SUKSESooo000ooo