FINAL TEST July 2011

Subject : Linear Algebra

Guidelines of the Answer!

1. Given $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

   (a). $P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = (-2 - \lambda)(1 - \lambda)(2 - \lambda) = -4 + 4\lambda + \lambda^2 - \lambda^3$

   (b). $\boldsymbol{A}$ haven't minimum polynomial

   (c). $P(\boldsymbol{A}) = 0 \rightarrow -4\boldsymbol{I} + 4\boldsymbol{A} + \boldsymbol{A}^2 - \boldsymbol{A}^3 = 0 \rightarrow 4\boldsymbol{I} = 4\boldsymbol{A} + \boldsymbol{A}^2 - \boldsymbol{A}^3 \rightarrow \boldsymbol{A}^{-1} = \frac{1}{4}(4\boldsymbol{I} + \boldsymbol{A} - \boldsymbol{A}^2)$

   $$\boldsymbol{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

2. Given $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

   $P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = \lambda(3 - \lambda)(3 - \lambda) = 9\lambda - 6\lambda^2 + \lambda^3$

   $P(\lambda) = 0 \rightarrow \lambda_1 = 0,\ \lambda_{2,3} = 3$

   For $\lambda = 0 \rightarrow X_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; For $\lambda = 3 \rightarrow X_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

   Then,

   $$W_1 = X_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

   Orthogonalization X_2 and X_3 :

   $W_2 = X_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $W_3 = X_3 - \frac{X_3.X_2}{X_2.X_2} X_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$

   Then normalize W_1, W_2, and W_3 :

$W_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \rightarrow Y_1 = \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$; $W_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \rightarrow Y_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$; and $W_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \rightarrow Y_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$

Orthogonal matrix $\boldsymbol{P} = (Y_1\ Y_2\ Y_3) = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \rightarrow \boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{P}^{T}\boldsymbol{AP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

3. $q = \frac{3}{2}\ p_1 - \frac{1}{2}\ p_2 - \frac{1}{2}\ p_3$, so $q_{\boldsymbol{P}} = (\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$.

4. Given $\boldsymbol{T} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \rightarrow P(\lambda) = |\boldsymbol{T} - \lambda\boldsymbol{I}| = (xt - zy) - (t + x)\lambda + \lambda^2$.

   (a) have two real and different eigen value : $(t + x)^2 - 4(xt - zy) > 0$

   (b) have one eigen value : $(t + x)^2 - 4(xt - zy) = 0$

   (c) have not real eigen value (having complex eigen value): $(t + x)^2 - 4(xt - zy) < 0$.

5. $T\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = \begin{pmatrix} x - y + 2z - 2s + t \\ 3x - 3y + 4z + 7s - t \\ -2x + 2y - 2z - 9s + 2t \\ 4x - 4y + 6z + 5s \end{pmatrix}$.

   (a). dim Im(T) = 3, basis = $\{\begin{pmatrix} 1 \\ 3 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}\}$

   (b). dim Ker(T) =, basis = $\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}\}$

6. Given $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 4 \\ 2 & 3 & 2 \\ -3 & -2 & -5 \end{pmatrix}$

$P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = (-2 - \lambda)(-1 - \lambda)(3 - \lambda) = 6 + 7\lambda - \lambda^3$

$P(\lambda) = 0 \rightarrow \lambda_1 = -2$, $\lambda_2 = -1$, and $\lambda_3 = 3$

For $\lambda = -2 \rightarrow X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; For $\lambda = -1 \rightarrow X_2 = \begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix}$, and $X_3 = \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}$.

Because the matrix ***A*** have three independence invariant vectors, ***A*** is diagonalizable.

$$\boldsymbol{P} = (X_1\ X_2\ X_3) = \begin{pmatrix} -1 & -6 & -2 \\ 0 & 1 & -5 \\ 1 & 4 & 2 \end{pmatrix} \rightarrow \boldsymbol{P}^{-1} = \frac{1}{10}\begin{pmatrix} 22 & 4 & 32 \\ -5 & 0 & -5 \\ -1 & -2 & -1 \end{pmatrix} \rightarrow \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

7. Find orthonormal basis of R^3, if one of the basis vectors is $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$!

Suppose : $X_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$, then take $X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow X_1.X_2 = 0 \rightarrow 2a - b - 2c = 0 \rightarrow X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Suppose we take $X_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \rightarrow X_3.X_2 = 0$ and $X_3.X_1 = 0$, so :

$$d + f = 0$$
$$2d - e - 2f = 0$$

From those equations: $X_3 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$

Normalize X_1, X_2, and X_3, it will got orthonormal basis :

$$Y_1 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}, Y_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \text{and } Y_3 = \begin{pmatrix} -\frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \end{pmatrix}.$$

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